Understandable Proofs of Unsatisfiability

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Pythagorean Triples Problem (I) [Ronald Graham, early 80's]

Will any coloring of the positive integers with red and blue result in a monochromatic Pythagorean Triple $a^2 + b^2 = c^2$?

$$
\begin{array}{cccc}
3^2 + 4^2 = 5^2 & 6^2 + 8^2 = 10^2 & 5^2 + 12^2 = 13^2 & 9^2 + 12^2 = 15^2 \\
8^2 + 15^2 = 17^2 & 12^2 + 16^2 = 20^2 & 15^2 + 20^2 = 25^2 & 7^2 + 24^2 = 25^2 \\
10^2 + 24^2 = 26^2 & 20^2 + 21^2 = 29^2 & 18^2 + 24^2 = 30^2 & 16^2 + 30^2 = 34^2 \\
21^2 + 28^2 = 35^2 & 12^2 + 35^2 = 37^2 & 15^2 + 36^2 = 39^2 & 24^2 + 32^2 = 40^2
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Best lower bound: a bi-coloring of $[1, 7664]$ s.t. there is no monochromatic Pythagorean Triple [Cooper & Overstreet 2015].

Myers conjectures that the answer is No [PhD thesis, 2015].

Pythagorean Triples Problem (II) [Ronald Graham, early 80's]

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A bi-coloring of $[1, n]$ is encoded using Boolean variables x_i with $i \in \{1, 2, \ldots, n\}$ such that $x_i = 1$ $(= 0)$ means that i is colored red (blue). For each Pythagorean Triple $a^2 + b^2 = c^2$, two clauses are added: $(x_a \vee x_b \vee x_c)$ and $(\overline{x}_a \vee \overline{x}_b \vee \overline{x}_c)$.

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Theorem ([Heule, Kullmann, and Marek (2016)])

$[1, 7824]$ *can be bi-colored s.t. there is no monochromatic Pythagorean Triple. This is impossible for* $[1, 7825]$.

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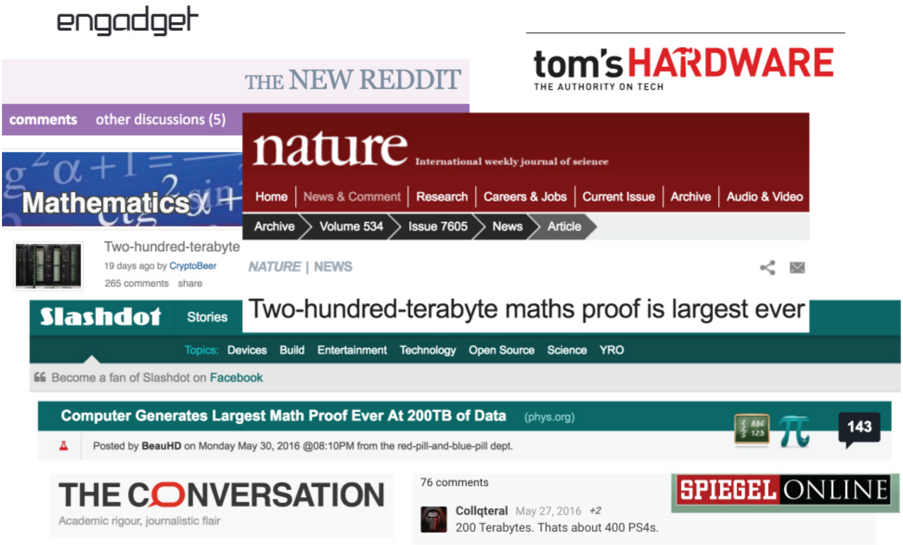
200 terabytes proof, but validated with verified checker

Media: “The Largest Math Proof Ever”



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Transition point: $n = 7\,825$

- Initial: $\#cls = 18\,944$, $\#var = 6\,495$, $\rho = 2.92$
- Preprocessing: $\#cls = 14\,672$, $\#var = 3\,746$, $\rho = 3.92$
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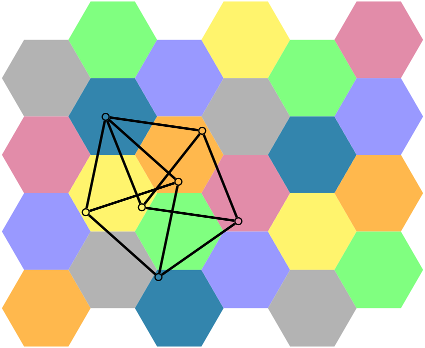
Way beyond transition point: $n = 1\,000\,000$

- Initial: $\#cls = 3\,961\,284$, $\#var = 866\,075$, $\rho = 4.57$
- Preprocessing: $\#cls = 635\,664$, $\#var = 64\,128$, $\rho = 9.91$
- Proof size: 1 terabyte

Chromatic Number of the Plane

The Hadwiger-Nelson problem:
How many colors are required to color the plane such that each pair of points that are exactly 1 apart are colored differently?



- The Moser Spindle graph shows the lower bound of 4
- A coloring of the plane showing the upper bound of 7

Small 5-Chromatic Unit Distance Graph

Use similar techniques:

- create a large, dense graph with chromatic number 5
- construct a small proof of unsatisfiability and extract core